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ORIGINAL PAPER

A contemporary linear representation theory for ordinary differential equations: probabilistic evolutions and related approximants for unidimensional autonomous systems

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Abstract In this paper, we build on our previous research on probabilistic foundations of dynamical systems and introduce a theory of linear representation for ordinary differential equations. The theory is developed for explicit ODEs and can be further extended to cover implicit cases. In this report, we investigate the case of a canonical single unknown autonomous system. First we construct a linear representation to get an infinite linear ODE set with a constant coefficient matrix which can be transformed into an upper triangular form. Then we find its approximate truncated solutions. We describe a number of properties of the theory using this framework. The companion of this paper expands this canonical approach to cover multidimensional cases using the theory of folded arrays which is another line of research established by our research group.

Keywords Dynamical systems · Probability · Expectation values · Ordinary differential equations · Linear algebra · Matrix theory

1 Introduction

This work is devoted to the investigation of the initial value problems of explicit ordinary differential equations (ODEs). Mathematical research on this topic is vast, therefore we cannot possibly justly reference all contributers to this massive topic.

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Having said that, we would like to point out the work by Arnold [1] which is relevant to the Evolution Operator concept we are going to present here. The primary focus of this paper is to introduce a new approach to linearization of a single explicit ODE with initial condition. ODEs are used in a diverse set of scientific and engineering disciplines with a variety of goals. One of the goals of our research is to introduce a linearization which feels more natural for certain disciplines such as quantum chemistry. The approach described here relies on our previous research in which we developed a first order infinite set of ODEs with an infinite constant coefficient matrix under an infinite number of initial conditions [2,3]. The most influential source of inspiration with this new approach is earlier research on the expectation dynamics in quantum mechanics [4,5] which we have further translationally developed to investigate dynamical systems [6–8]. Another source of motivation for this work, is the Lie algebraic factorization of the evolution operators which was proposed earlier by the first author [14,15]. Furthermore, our efforts are subtended by intuitions underlying the mathematical fluctuation concept and the fluctuationless theorem developed in the first author's group which has found contemporary applications on ODEs and related problems [9–13].

In this work, we follow established and pure mathematical principles to construct a set of probabilistic evolution equations using a basis set. We also construct the ODEs for the set elements. In this first paper, we detail the canonical mathematical structure underlying the application of this approach to the one unknown case to show the essential properties of our perspective. The more complicated case which involves multiple unknowns is basically an extension of the principles outlined in this paper to multidimensional arrays and folded arrays (folvec, folmat, folarr).

This paper is organized as follows. The second section presents the linear representation of a single autonomous ODE. In the third section, we explain the motivation behind using the term "Probabilistic Evolution". The fourth section is primarily about the triangular cases. Subsequently, in the fifth section, we describe the spectral properties for the triangular case. Then, the sixth section proceeds with elaborating on the explicit properties of the probabilistic evolution approximants. The seventh and eighth sections are devoted to the introduction of the "Evolution Operator" and the validation of the method in the context of analytic and numeric applications. Finally, the ninth section includes the concluding remarks for the paper.

2 Linear representation of a single autonomous ODE

Let us consider the following most general explicit one unknown autonomous ODE with the accompanying initial condition

$$\dot{\xi}(t) = f(\xi(t)), \qquad \xi(0) = a \tag{1}$$

where all entities are real valued. It is important to note that this is not a limitation especially since the case in which entities have complex values can be converted to a set of two ordinary differential equations with appropriate initial conditions by separately investigating the real and imaginary parts. In this sense, the method is not limited to real valued entities, however for the sake of simplicity in the description of the method, we will adhere to the case where all entities are real valued.

It is also important to note that contingencies of autonomy and explicitness can similarly be circumvented. A nonautonomous ODE can be converted to a set of autonomous ODEs by defining a new unknown equivalent to a first degree polynomial of the independent variable and related ODE. Such transformations underly a series of well established approaches in mathematics such as space extension and kernel methods.

In order to circumvent the contingency of explicitness, differential calculus with appropriate limitations can be used. In this report, we will not be further elaborating on how these transformations need to be conducted. We only refer to these approaches to show the vast applicability of this approach.

We also assume that the initial value a lies in the analyticity domain of the function $f(x)$ even though it seems to be possible to extend our analysis to the nonanalytical cases with some precautions. If we consider a reference point that is represented by x_{ref} which lies in the analyticity domain of f then we can write the following Taylor series expansion

$$f(x) = \sum_{j=0}^{\infty} f_j \left(x - x_{ref}\right)^j \tag{2}$$

where the coefficient f_j stands for the value of the jth derivative of f at $x = x_{ref}$ divided by $j!$ (2) implies that the right hand side of (1) is an infinite linear combination of the power functions $\left(\xi(t) - x_{ref}\right)^j$ for natural number values of j. These power functions are in fact the basis functions of the Taylor expansion which urges us to construct an ODE for each of these basis functions. We can first define

$$x_j(t) \equiv \left(\xi(t) - x_{ref}\right)^{j-1}, \quad j = 1, 2, 3, \ldots \tag{3}$$

and write

$$\begin{aligned} \dot{x}_j(t) &= (j-1)\left(\xi(t) - x_{ref}\right)^{j-2} \dot{\xi}(t) = (j-1)\left(\xi(t) - x_{ref}\right)^{j-2} f\left(\xi(t)\right) \\ &= \sum_{k=0}^{\infty} (j-1) f_k \left(\xi(t) - x_{ref}\right)^{k+j-2} = \sum_{k=0}^{\infty} (j-1) f_k x_{k+j-1}(t), \quad j = 1, 2, 3, \ldots \end{aligned} \tag{4}$$

which can be put into the following more amenable matrix algebraic form by adding the corresponding initial conditions

$$\dot{\mathbf{x}}(t) = \mathbf{E}\mathbf{x}(t), \qquad \mathbf{x}(0) = \mathbf{a} \tag{5}$$

where

$$\mathbf{x}(t) = \left[x_1(t)\ x_2(t)\ x_3(t) \ldots\right]^T, \qquad \mathbf{a} = \left[1\ \left(a - x_{ref}\right)\ \left(a - x_{ref}\right)^2\ \ldots\right]^T \tag{6}$$

and

$$\mathbf{E} = \begin{bmatrix} 0 & 0 & 0 & 0 & \dots \\ f_0 & f_1 & f_2 & f_3 & \dots \\ 0 & 2f_0 & 2f_1 & 2f_2 & \dots \\ 0 & 0 & 3f_0 & 3f_1 & \dots \\ 0 & 0 & 0 & 4f_0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}. \tag{7}$$

We call $\mathbf{E}$ the "Evolution Matrix" since it is critical for specifying the evolution (in time) of the sytem characterized by the infinite temporally varying state vector $\mathbf{x}(t)$. More explicitly, the first row of the Evolution matrix is composed of zeroes while the second row elements are the Taylor series expansion coefficents. The third row of the Evolution Matrix is the one element rightward shifted form of the second row after multiplication by 2 and addition of 0 as the first element. The nth row starts with $(n-1)$ zero elements and then respectively contains the second row elements multiplied by $(n-1)$. In other words, the Evolution Matrix is in an upper Hessenberg form.

The equation in (5) is the initial value problem of an infinite set of linear ordinary differential equations. The second element of its solution vector gives the sought function $\xi(t)$ after increasing the value by x_{ref} as the solution of (1). The linearity facilitates the analysis. However we have to be careful about potential convergence issues coming from the infinite structures of the considered entities, the Evolution Matrix $\mathbf{E}(t)$, the state vector $\mathbf{x}(t)$ and the initial vector $\mathbf{a}$. The formal solution of (5) can be written as follows

$$\mathbf{x}(t) = \mathrm{e}^{t\mathbf{E}}\mathbf{a} \tag{8}$$

where the evaluation of the infinite exponential matrix function becomes the focal issue. Even though a number of methods are applicable in this context, we prefer to use the approach of spectral representation. However this requires the solution of the eigenvalue problem of an infinite matrix which may be nontrivial. This is the case, especially since it is possible for a continuous spectrum to emerge, making the corresponding eigenvectors non-convergent standard vectors (this may happen when it is not possible to construct converging infinite sequences of finite truncations to ODE). On the other hand, the case where f_0 vanishes substantially facilitates the analysis since the Hessenberg form of the Evolution Matrix turns out to be a triangular matrix. The spectrum of this triangular matrix is composed of single eigenvalues which are proportional to f_1 at scales of natural numbers. The lack of the multiplicity facilitates the analysis since it precludes cases where the algebraic and geometric multiplicities of certain eigenvalues are different, which would make the matrix potentially non-diagonalizable. The vanishing f_0 corresponds to the case where the right hand side function f vanishes at $x = x_{ref}$. If this happens, then the initial condition $x(0) = x_{ref}$ enforces the solution, the function $\xi(t)$, to remain at the constant value x_{ref}. In other words, the solution of (1) gets positioned at the phase space point where ξ and its temporal derivatives take the values x_{ref} and 0 respectively when the initial value a, moving along a path passing through x_{ref}, arrives at x_{ref}.

The circumstances described above can arise only when f has at least one zero. There is no warranty for the existence of at least one zero to the function f and we disregard singularities of f. In the interest of simplicity we confine ourselves to cases where the function f is analytic (equivalently holomorphic) everywhere in the complex plane of x. In other words, the complex plane is considered to be composed of only finite complex number pairs. Its each point uniquely corresponds to a point on the Riemann sphere whose equator is the unit circle (the circle with a radius equal to 1, centered at the origin of the complex plane), on the complex plane. The line segment which joins a point in the complex plane to the north pole of the Riemann sphere intersects with the Riemann sphere. This correspondence between the intersection points and the complex plane points spans all points of the Riemann sphere except its north pole which is used as the reference point to make correspondence between the sphere and complex plane points.

The complex plane points remaining inside its unit circle correspond to the lower (south) Riemannian hemisphere while the upper (north) Riemannian hemisphere points are corresponding to the complex plane points residing outside its unit circle. Each unit circle point of the complex plane remains at the same position but on the equator of the Riemann sphere. The intersection point can not be located on the north pole unless we do not take the line segment in a plane parallel to the complex plane. Since this parallel plane can intersect with the complex plane only at infinity, any infinite complex number can be considered to be represented by the Riemann sphere's north pole. Hence this north pole is identified as "Complex Infinity ". The addition of the north pole to the Riemann sphere corresponds to the addition of the Complex Infinity to the complex plane. The formed plane is called "Extended Complex Plane". An entire function remains analytic everywhere except Complex Infinity. However, this analyticity does not require the existence of zeroes for the entire function under consideration. If it has zeroes then each zero enables us to get triangular Evolution Matrix by taking the value x_{ref} as the considered zero. This means that triangularity can be obtained only at the zeroes of f. All other points enforce us to use upper Hessenberg forms.

These discussions urge us to explore other way(s) to work with the right hand side functions. To this end, we can use the so-called method, Space Extension". We assume that $f(x)$, which is assumed to be entire, in (1) never vanishes except at Complex Infinity (it is very well known that an entire function can take all values residing in the complex plane with the unlikely exception of a single point such as zero in this case; exponential function forms a good example to this end). We can define

$$\eta(t) \equiv f\left(\xi(t)\right) \tag{9}$$

and get the following ODE by simple temporal differentiation

$$\dot{\eta}(t) = f'\left(\xi(t)\right)\dot{\xi}(t) = f'\left(\xi(t)\right) f\left(\xi(t)\right) = \eta(t) f'\left(\xi(t)\right) \tag{10}$$

where prime stands for the differentiation with respect to argument. After the definition in (9) we can rewrite the ODE in (1) as follows

$$\dot{\xi}(t) = \eta(t). \tag{11}$$

Now we can gather what we have obtained here, in the following set of ODEs and accompanying initial conditions

$$\begin{aligned} \dot{\xi}(t) &= f_1\left(\xi(t), \eta(t)\right), & \xi(0) &= a; \\ \dot{\eta}(t) &= f_2\left(\xi(t), \eta(t)\right), & \eta(0) &= f(a), \end{aligned} \tag{12}$$

where

$$f_1\left(\xi, \eta\right) \equiv \eta, \qquad f_2\left(\xi, \eta\right) = f'\left(\xi\right)\eta. \tag{13}$$

The Eq. (12) depend on the functional structure of $f(x)$ not only through the ODEs but also their initial conditions. The functional structure in the ODE part is now changed. In the companion paper we focus on other forms of space extension which reduce the ODE functional structure over the extended space to an extremely simple form which has entire function structures even if the original ODE has singularities.

Equation (12) can be obtained for ODEs which are translated by a constant (i.e., $f(x)+C$ where C is a constant). The translation can make it possible for the right hand side to attain the vanishing property. This is reflected in the equations in (12) as vanishing properties. The equations in (12) contain two unknowns. In other words, since there are two state functions ξ and η the dimension of the state space is increased by one. This is the reason why we engage in "Space Extension". The companion of this paper focuses on the multidimensional cases by covering certain space extension possibilities.

The functions in (13) vanish on the ξ axis of (ξ, η)–plane. This means that the constant vector value at the right hand side vanishes. In this multidimensional case, we can use the folded matrix (folmat) concept. The Evolution Folmat of this case has hypertriangular form and hence the spectral investigation becomes quite easy.

The space extension approach does not only extend the space to higher dimensions. It also extends the problem under consideration to a broader class. This extension is not only for adding vanishing properties to the right hand side function(s). It can vastly simplify the structure of the Evolution Folmat over the extended space. The first author of the paper reported a method to get conical structure (second degree multinomial) in the right hand side function(s) [14,15]. We do not intend to focus on these kinds of simplifications. For the purposes of this paper, we find it sufficient to utilize triangular structures.

3 Why probabilistic evolution?

The Eq. (8) describes the motion of a hypothetical point in an infinite dimensional Cartesian space. The equation is in a particular form due to the nature of the initial vector $\mathbf{a}$ whose elements are natural number powers of a single parameter, $\left(a - x_{ref}\right)$. We call this type of initial vectors “Single Parameter Power Vector”. This specific power nature of the initial vector in (8) is an important limitation even in the case of a single ODE. The ODE in (5) can be more generalized by extending the initial conditions to the infinite vector whose elements are certain arbitrary values, $a_0, a_1, a_2, \ldots$. Even though these are not powers of a single parameter, it is possible to consider them in an initial vector which is finite or infinite linear combinations of certain power vectors with different parameter values. Therefore, we can write

$$a_j \equiv \int_{-\infty}^{\infty} d\alpha W(\alpha)\alpha^j, \qquad j = 0, 1, 2, \ldots \tag{14}$$

where $W(\alpha)$ needs be a true weight function, at least in a finite interval even though discrete weights can be separately investigated, otherwise certain complications causing some level of arbitrarinesses may occur in the sense of probabilistic nature. This urges us to write

$$x_j(t) \equiv \int_{-\infty}^{\infty} d\alpha W(\alpha) y_j(t, \alpha), \qquad j = 0, 1, 2, \ldots \tag{15}$$

which results in an infinite ODE and accompanying initial conditions like in (5). This suggests the use of expectation values for expressing the initial vector and therefore the unknowns which are expressed as temporally changing expectation values for the elements of $\mathbf{x}(t)$ vector. The introduction of expectation values brings the probability concept to the scene. Therefore a single power vector is related to the Dirac delta function type distributions which characterize the probability. In other words, the case in (8) corresponds to sharply localized probability densities. All these facts urge us to call (5) “Probabilistic Evolution Equation (PEE)”. The evolution of the system in time is characterized by $\mathbf{E}$ and the propagation of the system state independent of the initial condition is described by the exponential matrix $e^{t\mathbf{E}}$ we call “Probabilistic Matrix Propagator” or simply “Matrix Propagator”.

It is important to emphasize the difference between the linear independence and the functional independence. What we have used here is the linear independence of the basis functions $x_j(t)$s. This linear independence enables them to span an appropriate linear vector space which may become a Hilbert space under an appropriate inner product definition. On the other hand, these functions are functionally dependent because they are different instantiations of a single canonical form. The use of linear independence takes us to completely linear ODEs on an appropriate infinite linear vector space while the functional dependence preserves the nonlinearity and therefore the original concise structure. Finally, it is important to point out “The use of linear independence over infinite dimensional space underlies linearity”, on the other hand, “Finiteness in the working space is attained at the expense of nonlinearity”. This may be considered as a quite general and valid “rule of thumb”. Kernel space methods developed for nonlinear data processing can be given as good examples of this intuition [16,17].

4 Triangular case and triangular approximants

The discussions above are only relevant for cases where f_0 vanishes. The Evolution Matrix $\mathbf{E}$, which is a function of x_{ref}, becomes triangular for these cases. This triangularity is quite important since it enables us to evaluate the exponential of the Evolution Matrix through finite truncations via an easy way such that the differences from one truncation to its nearest higher neighbor occur in only extended rows and columns. To explain what happens we can define

$$\mathbf{E}_n \equiv \begin{bmatrix} 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & f_1 & f_2 & f_3 & \dots & f_n \\ 0 & 0 & 2f_1 & 2f_2 & \dots & 2f_{n-1} \\ 0 & 0 & 0 & 3f_1 & \dots & 3f_{n-2} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & nf_1 \end{bmatrix} \qquad \mathbf{a}_n \equiv \begin{bmatrix} 1 \\ \left(a - x_{ref}\right) \\ \left(a - x_{ref}\right)^2 \\ \vdots \\ \left(a - x_{ref}\right)^n \end{bmatrix} \tag{16}$$

$$\theta_n(t, a) \equiv \mathbf{e}_2^{(c,n+1)^T} \mathrm{e}^{t\mathbf{E}_n} \mathbf{a}_n, \qquad n = 0, 1, 2, \dots \tag{17}$$

where $\mathbf{e}_2^{(c,n+1)}$ stands for the second standard unit vector of $(n+1)$-dimensional Cartesian space, that is, the unit vector whose only nonzero element which takes the value of 1 resides on the second position.

It is not hard to show that the following relation,

$$\mathrm{e}^{t\mathbf{E}_{n+1}} = \begin{bmatrix} \mathrm{e}^{t\mathbf{E}_n} & \left(\mathrm{e}^{t\mathbf{E}_n} - \mathrm{e}^{(n+1)f_1 t}\mathbf{I}_{n+1}\right)\left[\mathbf{E}_n - (n+1)f_1\mathbf{I}_{n+1}\right]^{-1}\mathbf{u}_{n+1} \\ \mathbf{0}_{n+1}^T & \mathrm{e}^{(n+1)f_1 t} \end{bmatrix}$$
$$n = 0, 1, 2, \dots, \tag{18}$$

for $(n+1) \times (n+1)$ type identity matrix $\mathbf{I}_{n+1}$ and

$$\mathbf{0}_{n+1}^T \equiv [\,0\ \dots\ 0\,], \qquad \mathbf{u}_{n+1} \equiv \left[\,0\ f_{n+1}\ 2f_n\ 3f_{n-1}\ \dots\ nf_2\,\right] \tag{19}$$

(where $\mathbf{0}_{n+1}$ contains $(n+1)$ number of zeroes) hold. This means that when we pass from the nth truncation approximant to the $(n+1)$th one, the first $(n+1)$ terms do not change and the change is proportional to $\left(a - x_{ref}\right)^{n+1}$. This is a beautiful property since all coefficients of $\left(a - x_{ref}\right)$ powers existing in an approximant never changes in each step from another approximant, whose order is greater than the considered approximant, to the next higher ordered one. As a matter of fact this corresponds to extracting the factor $\left(a - x_{ref}\right)$ from the unknown in the original ODE and then expanding the unknown to powers of $\left(a - x_{ref}\right)$, and finally, truncating the expansion at the order of the truncation approximant of the probabilistic evolution (this is closely related to the Taylor approximation method for ODEs when the approximants are expanded in powers of t).

The triangular structure of $\mathbf{E}_n$ enables us to evaluate the inversion term $\left[\mathbf{E}_n - (n+1)f_1\mathbf{I}_{n+1},\right]^{-1}$ in a rather easy way. It even permits us to construct a recursive method to accomplish this. Hence, the right uppermost term in (18) can be evaluated recursively since the exponential factor $e^{t\mathbf{E}_n}$ is assumed to be known. Therefore the implementation of this procedure is straightforward. Therefore we can comfortably say that the truncation approximants of probabilistic evolution can be evaluated rather easily. It fixes the first $(n+1)$ terms of the true solution of the infinite element case after the construction of the nth approximant.

We call $\theta_n(t,a)$ which is an nth degree polynomial in $\left(a - x_{ref}\right)$ "nth Probabilistic Evolution Approximant". This approximant matches the nth degree polynomial truncation of $\theta_\infty(t,a)$ in $\left(a - x_{ref}\right)$ because of the triangularity in the Evolution Matrix.

5 Spectral issues for the triangular case

Now we focus on the spectral properties of the Evolution Matrix $\mathbf{E}$ for the case of vanishing f_0. The triangular structure of $\mathbf{E}$ can be expressed by using the following block representations

$$\mathbf{E} \equiv \begin{bmatrix} \mathbf{E}_{n-1} & \mathbf{u}_n & \mathbf{R}_n \\ \mathbf{0}_n^T & nf_1 & \mathbf{v}_n^T \\ \mathbf{0}_{\infty\times n} & \mathbf{0}_\infty & \mathbf{E}_{(n+1)\rightarrow\infty} \end{bmatrix} \tag{20}$$

where $\mathbf{E}_{n-1}$, $\mathbf{0}_n$ and $\mathbf{0}_\infty$ stand for the left uppermost $n \times n$ truncation of the Evolution matrix as we used before and the zero matrices with n and infinite number of elements respectively while $\mathbf{0}_{\infty\times n}$ denotes the zero matrix composed of n infinite columns. While the n element vector $\mathbf{u}_n$ is same as its previously defined form, the other blocks are defined as follows

$$\mathbf{R}_n \equiv \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots \\ f_{n+1} & f_{n+2} & f_{n+3} & f_{n+4} & \cdots \\ 2f_n & 2f_{n+1} & 2f_{n+2} & 2f_{n+3} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \\ (n-1)f_3 & (n-1)f_4 & (n-1)f_5 & (n-1)f_6 & \cdots \end{bmatrix} \tag{21}$$

$$\mathbf{v}_n^T \equiv [\, nf_2 \; nf_3 \; nf_4 \; \ldots \,] \tag{22}$$

$$\mathbf{E}_{(n+1)\rightarrow\infty} \equiv \begin{bmatrix} (n+1)f_1 & (n+1)f_2 & (n+1)f_3 & (n+1)f_4 & \cdots \\ 0 & (n+2)f_1 & (n+2)f_2 & (n+2)f_3 & \cdots \\ 0 & 0 & (n+3)f_1 & (n+2)f_2 & \cdots \\ 0 & 0 & 0 & (n+4)f_1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \tag{23}$$

The eigenvalue problem of the Evolution Matrix can be formulated as follows

$$\mathbf{E}\mathbf{e}_k^{(r)} = \epsilon_k \mathbf{e}_k^{(r)}, \qquad \mathbf{E}^T \mathbf{e}_k^{(\ell)} = \epsilon_k \mathbf{e}_k^{(\ell)}, \qquad k = 1, 2, 3, \ldots \tag{24}$$

where ϵ_k stands for the kth eigenvalue while $\mathbf{e}_k^{(r)}$ and $\mathbf{e}_k^{(\ell)}$ denote the corresponding right and left eigenvectors respectively. The eigenvectors are apparently given by the following equalities

$$\epsilon_k = kf_1, \qquad k = 0, 1, 2, \ldots \tag{25}$$

while the following dual orthonormality conditions are satisfied amongst the right and left eigenvectors

$$\left(\mathbf{e}_j^{(\ell)}, \mathbf{e}_k^{(\ell)}\right) \equiv \mathbf{e}_j^{(\ell)^T} \mathbf{e}_k^{(\ell)} = \delta_{j,k}, \qquad j, k = 1, 2, \ldots \tag{26}$$

where $\delta_{j,k}$ stands for the Kroenecker's delta symbol which vanishes for different j, k values while it becomes 1 when j, k match.

Because of the symmetry in first row and column the zeroth eigenvectors, left and right, are all $\mathbf{e}_1$ which is first unit vector of the Cartesian space. They correspond to the zeroth eigenvalue which vanishes. For the remaining cases, the explicit block structures of the right and left eigenvectors can be given as follows by skipping intermediate details of the routine derivation procedure

$$\mathbf{e}_n^{(r)} = \begin{bmatrix} -\left[\mathbf{E}_{n-1} - \epsilon_n \mathbf{I}_n\right]^{-1} \mathbf{u}_n \\ 1 \\ \mathbf{0}_\infty \end{bmatrix}, \qquad n = 1, 2, 3, \ldots \tag{27}$$

$$\mathbf{e}_n^{(\ell)} = \begin{bmatrix} \mathbf{0}_n \\ 1 \\ -\left[\mathbf{E}_{(n+1)\to\infty}^T - \epsilon_n \mathbf{I}_\infty\right]^{-1} \mathbf{v}_n \end{bmatrix}, \qquad n = 1, 2, 3, \ldots \tag{28}$$

6 Explicit properties of probabilistic evolution approximants

To get more explicit formulae for probabilistic evolution approximants we can start by explicitly giving the following spectral representation

$$e^{t\mathbf{E}_n} = \sum_{j=0}^{n} e^{jf_1t} \mathbf{e}_{j+1}^{(r,n+1)} \mathbf{e}_{j+1}^{(\ell,n+1)^T} \tag{29}$$

where the superscripts $(r, n+1)$ and $(\ell, n+1)$ refer to right and left eigenvectors of $\mathbf{E}_n$, which is the $(n+1)$ dimensional truncation of the Evolution Matrix defined over infinite dimensional Cartesian space, respectively. This representation enables us to write

$$\theta_n(t, a) = \sum_{j_1=0}^{n} \sum_{j_2=0}^{n} \Theta_{j_1+1, j_2+1}^{(n)} e^{j_1 f_1 t} \left(a - x_{ref}\right)^{j_2} \tag{30}$$

where, $\mathbf{e}_j^{(c,n+1)}$ being the jth standard unit vector of the $(n+1)$ dimensional Cartesian space, which has 1 as only nonzero element located at its jth place,

$$\Theta_{j_1+1, j_2+1}^{(n)} \equiv \left(\mathbf{e}_2^{(c,n+1)^T} \mathbf{e}_{j_1+1}^{(r,n+1)}\right) \left(\mathbf{e}_{j_1+1}^{(\ell,n+1)^T} \mathbf{e}_{j_2+1}^{(c,n+1)}\right), \quad j_1, j_2 = 0, \ldots, n. \tag{31}$$

Apparently the matrix $\boldsymbol{\Theta}_n$ is upper triangular because of the particular natures of the left eigenvectors. To proceed, we can define the power vector as follows

$$\mathbf{p}(x) \equiv \left[1 \; x \; x^2 \ldots \right]^T \tag{32}$$

which is a one parameter infinite vector. Its finite truncations can also be used when they are needed. Now (30) can be rewritten in the following concise form

$$\theta_n(t,a) = \mathbf{p}_n\left(\mathrm{e}^{f_1 t}\right)^T \boldsymbol{\Theta}_n \mathbf{a}_n \tag{33}$$

where $\mathbf{p}_n$ means a power vector truncated at the nth power. The upper triangularity in $\boldsymbol{\Theta}_n$ enables us to construct the following recursion over θ_ns

$$\theta_n(t,a) = \theta_{n-1}(t,a) + \sum_{j_1=0}^{n} \Theta^{(n)}_{j_1+1,n+1} \mathrm{e}^{j_1 f_1 t}\left(a - x_{ref}\right)^n \tag{34}$$

or more explictly

$$\begin{aligned}\theta_n(t,a) &= \theta_{n-1}(t,a) \\ &\quad + \sum_{j_1=0}^{n}\left(\mathbf{e}_2^{(c,n+1)^T}\mathbf{e}_{j_1+1}^{(r,n+1)}\right)\left(\mathbf{e}_{j_1+1}^{(\ell,n+1)^T}\mathbf{e}_{n+1}^{(c,n+1)}\right)\mathrm{e}^{j_1 f_1 t}\left(a - x_{ref}\right)^n\end{aligned} \tag{35}$$

by skipping the intermediate construction details.

Upper triangular matrix $\boldsymbol{\Theta}_n$'s first row is completely composed of zeroes. On the other hand, all elements of $\boldsymbol{\Theta}_n$ are same as the elements of the same locations in all $\boldsymbol{\Theta}_m$ matrices for m values greater than n. First few nonzero elements of these matrices are given below

$$\begin{aligned}&\Theta^{(n)}_{1,1} = 1, \quad \Theta^{(n)}_{1,2} = -\frac{f_2}{f_1}, \quad \Theta^{(n)}_{1,3} = -\frac{f_3}{2f_1} + \frac{f_2^2}{f_1^2}, \quad \Theta^{(n)}_{1,4} = -\frac{f_4}{3f_1} + \frac{7f_2f_3}{6f_1^2} - \frac{f_2^3}{f_1^3} \\ &\Theta^{(n)}_{2,2} = \frac{f_2}{f_1}, \quad \Theta^{(n)}_{2,3} = -\frac{2f_2^2}{f_1^2}, \quad \Theta^{(n)}_{2,4} = \frac{3f_2^3}{f_1^3} - \frac{f_2f_3}{f_1^2} \\ &\Theta^{(n)}_{3,3} = \frac{f_2^2}{f_1^2} + \frac{f_3}{2f_1}, \quad \Theta^{(n)}_{3,4} = -\frac{3f_2^3}{f_1^3} - \frac{3f_2f_3}{2f_1^2}, \\ &\Theta^{(n)}_{4,4} = \frac{f_2^3}{f_1^3} + \frac{4f_1f_2f_3}{3f_1^3} + \frac{f_4}{3f_1}.\end{aligned} \tag{36}$$

which remain valid as long as n is not less than anyone of the subindices.

We need to emphasize that all the analyses above remain valid as long as f_1 does not vanish. Otherwise a slightly different procedure should be used since all the eigen spectrum contracts to the single eigenvalue which vanishes. The vanishing zero eigenvalue is infinitely multiple and may require a Jordan canonical form type structure or otherwise certain direct algebraic analysis may need to be conducted to evaluate infinite or truncated matrix propagators.

7 Evolution operator

Let us consider a function $g(x)$ which can be expanded to a Taylor series as follows

$$g(x) \equiv \sum_{j=0}^{\infty} g_j \left(x - x_{ref}\right)^j \tag{37}$$

which can be represented in the following more comprehensible form

$$g(x) = \mathbf{g}^T \mathbf{p} \left(x - x_{ref}\right) \tag{38}$$

where

$$\mathbf{g} \equiv [\, g_0 \; g_1 \; \ldots \,]^T . \tag{39}$$

Now it is not hard to see that the following equalities,

$$\begin{aligned} \mathbf{Ep}\left(x - x_{ref}\right) &= \left(\sum_{k=0}^{\infty} f_k \left(x - x_{ref}\right)^k\right) \left[\, 0 \; 1 \; 2\left(x - x_{ref}\right) \; 3\left(x - x_{ref}\right)^2 \; \ldots \,\right]^T \\ &= \left(\sum_{k=0}^{\infty} f_k \left(x - x_{ref}\right)^k\right) \frac{\partial}{\partial x} \left[\, 1 \; \left(x - x_{ref}\right) \; \left(x - x_{ref}\right)^2 \; \ldots \,\right]^T , \end{aligned} \tag{40}$$

hold as long as the infinite series in the factor of the right hand side vectors converge. This convergence exists because of the analiticity in $f(x)$ and therefore we can write

$$\mathbf{Ep}\left(x - x_{ref}\right) = f(x) \frac{\partial}{\partial x} \left[\, 1 \; \left(x - x_{ref}\right) \; \left(x - x_{ref}\right)^2 \; \ldots \,\right]^T . \tag{41}$$

which implies

$$\begin{aligned} \mathbf{g}^T \mathbf{Ep}\left(x - x_{ref}\right) &\equiv \left(\mathbf{E}^T \mathbf{g}\right)^T \mathbf{p}\left(x - x_{ref}\right) = f(x) \frac{\partial}{\partial x} \mathbf{g}^T \mathbf{p}\left(x - x_{ref}\right) \\ &= f(x) \frac{\partial}{\partial x} g(x) \end{aligned} \tag{42}$$

This result shows that the action of the Evolution Matrix Transpose on an arbitrary analytic function $g(x)$'s coefficient vector can be represented by the operator whose action on its operand is differentiation followed by the multiplication with the function $f(x)$. Hence this operator's Hermitian conjugate should be corresponding to the Evolution Matrix. We define

$$\mathcal{E}^{\dagger} g(x) \equiv f(x) \frac{dg(x)}{dx} \tag{43}$$

where $g(x)$ denotes an arbitrary analytic function. We call $\mathcal{E}$ "Evolution Operator". Therefore the Evolution Matrix $\mathbf{E}$ represents the matrix representation of the Evolution Operator. It may help to investigate many issues over the continuous items which we plan to do so in our future works. Having said that, we may mention a property of $\mathcal{E}^{\dagger}$ here. It is very well known that the eigenvalues of $\mathcal{E}^{\dagger}$ match the eigenvalues of $\mathcal{E}$. On the other hand, the first order homogeneous structure of $\mathcal{E}^{\dagger}$ in the differentiation operator with respect to x enables us to write the following identities

$$\mathcal{E}^{\dagger} \left(g_1(x) g_2(x) \right) \equiv \mathcal{E}^{\dagger} \left(g_1(x) \right) g_2(x) + g_1(x) \mathcal{E}^{\dagger} \left(g_2(x) \right) \tag{44}$$

$$\mathcal{E}^{\dagger} \left(g(x)^m \right) \equiv m \mathcal{E}^{\dagger} \left(g(x) \right)^{m-1} \tag{45}$$

where g_1 and g_2 stand for arbitrary analytical functions while m takes natural number values. We can specify $\mathcal{E}^{\dagger}$'s eigenfunction corresponding to the eigenvalue f_1 by using the $\varphi(x)$ symbol. Then (45) means that the natural number powers of $\varphi(x)$ are also $\mathcal{E}^{\dagger}$'s eigenfunctions corresponding to different eigenvalues. More specifically, $\varphi(x)^n$ is the eigenfunction of $\mathcal{E}^{\dagger}$ corresponding to its nth eigenvalue, $\epsilon_n = n f_1$, for natural number values of n. Since the eigenvalues of $\mathcal{E}^{\dagger}$ are all single (whose multiplicities are just 1), the eigenfunction set of $\mathcal{E}^{\dagger}$ is spanned by the natural number powers of $\varphi(x)$. Although these eigenfunctions are linearly independent they are functionally dependent and generated by a single function $\varphi(x)$. Finding $\varphi(x)$ and its utilization in convergence analysis is left to one of our future works.

8 Analytical and/or numerical validations

The formal analytic solution of (1) can be obtained by solving t in terms of ξ as follows

$$t = \tau(\xi) = \int_a^{\xi} \frac{d\xi_1}{f(\xi_1)} \tag{46}$$

If the integration above cannot be analytically performed certain quadrature based approximation approaches that are in accordance with the structure of the integrand can be used. It is important to note that the equality in (46) is not the final result since we desire to obtain the explicit dependence of ξ on t. We need the inversion of the function τ since we can formally write

$$\xi = \tau^{-1}(t) \tag{47}$$

whose validity may not cover the whole nonnegative values of t. The function $f(\xi)$ and the initial parameter a are the agents that determine the validity domain. Even in the case of entire functions, inversion may not cover the whole domain (nonnegative values) of t. Nevertheless, there is an abundance of cases where this inversion can be realised analytically. In our experiments, we have observed that the analytical result is harmonious with the truncation approximants around the initial point. The Taylor series coefficients of the true solution in powers of t are revealed systematically by the truncation approximants as we increase the truncation order. Accordingly anticipated observations can be considered to signal the revelation of convergence properties.

9 Concluding remarks

We have developed an approximation scheme for the solution of a single autonomous explicit ordinary differential equation by using its probabilistic evolution equation and related truncation approximants. We enumerate important concluding remarks below

1. The first and most important original finding here is the construction of the probabilistic evolution equations by using expansions over basis functions;
2. Even though one unknown ODEs can be considered trivial, we extract a number of important general features which can be applied to ODEs with more than one unknowns;
3. We have focused on the cases where the right hand side function of the original ODE is an entire function which has at least one zero. If it has no zero except the one at complex infinity then space extension seems to be a practical and efficient method. On the other hand the existence of more than one zeroes may result in diverging infinite element vectors. The analytic continuation and space extension possibilities can again be considered as spare tools. The utilization of these items in the format of this paper is completely new. Especially triangularity in the evolution matrix is very important;
4. Even though we have not explicitly given the numerical experimentations on the entire right hand side functions having more than one zeroes, there are implications that the truncation approximants converge for some finite t intervals over which the inverse of $\tau(\xi)$ can be defined. These facts illuminate the road to finding convergence properties and probably brings the analytic continuation possibilities to mind. All these issues will be at the focus of our upcoming works;
5. It is important to consider cases where both f_0 and f_1 simultaneously vanish. We have the triangular structure again for this case. However all the diagonal elements of the Evolution Matrix vanish. Therefore, there, only a single but infinitely multiple eigenvalue of zero value is obtained. The infinite multiplicity results in Jordan canonical form since only two linearly independent eigenvectors are obtained for this case if f_2 does not vanish (otherwise the number of linearly independent eigenvectors increases depending on the number of vanishing first Taylor coefficients). The hidden infinitely many eigenvectors can be revealed as usual by seeking the eigenvectors of the powers of the Evolution Matrix. The Evolution Operator counterpart of the analysis in this case may help to reveal the important details even though it is left for future studies;
6. The entirety is not a great limitation and all we have done here can be modified for the singular cases. However, as we mentioned in this paper, the space extension concept may be useful to remove singularities from the right hand side function(s)

of ODE(s) at the expense of their appearences at the initial conditions. We do not focus on these issues in details here.

The companion of this paper focuses on space extension possibilities and extension of what we have done here to more than one unknown in the ODEs and accompanying initial conditions. We also give certain details of the multilinear algebra based on folded items, folvecs (folded vectors), folmats (folded matrices), folarrs (folded arrays) there.

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